The non-chiral fusion rules in rational conformal field theories.

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Abstract

We introduce a general method to construct the non-chiral fusion rules in rational conformal field theories. We are particularly interested by the models of the complementary series or like-D series which are solutions of modular invariant partition function. The form proposed of the non-chiral fusion rules has a structure of Z_n grading.

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1 Introduction

The study of the two-dimensional conformal field theories (CFT's) has flourished in recent years. Indeed, these theories proved to be useful for studying fundamental states of strings, in string theories and classifying the second order phase transitions in condensed matter physics. Also the richness of its formalism gave it a mathematical interest.

The symmetry algebra of a conformal field theory defined on manifold without boundary has a direct sum form : $\mathcal{A}\oplus\overline{\mathcal{A}}$, where $\mathcal{A}$ and $\overline{\mathcal{A}}$ are two copies of a chiral algebra, having the Virasoro algebra as sub-algebra [1]. These copies act respectively on the holomorphic (z) and antiholomorphic ($\overline{z}$) dependencies of the physical fields of the theory. Well-known examples of chiral algebras which are Lie algebras are: Virasoro algebra, Kac-Moody algebra and more generally the so-called W-algebra [2]. We assume at this point that the algebra of the holomorphic and antiholomorphic sectors of the theory are identical.

The physical Hilbert space is obtained as sum of irreducible representation spaces of chiral algebras. This is done by acting (with the universal enveloping algebra $\mathfrak{U}(\mathcal{A})$ of the Lie algebra ($\mathcal{A}$)) on a lowest-weight vector, and then projecting out null states. Via the correspondence between field and state vectors in physical Hilbert space of the theory, fields corresponding to lowest-weight vectors are referred to as primary fields, while those corresponding to non lowest-weight vectors are called descendant fields. If we denote $\mathfrak{L}_{\mathcal{A}}$ the list of all irreducible lowest-weight representation of $\mathcal{A}$, a chiral algebra ($\mathcal{A}$) is called rational if $\mathcal{A}$ is $*-$algebra and contains a finite set $\mathfrak{L}_{\mathcal{A}}$. A conformal field theory underlying a rational chiral algebra is *rational conformal field theory* (RCFT). RCFT are particularly interesting, however these theories take their designation from the fact that the chiral algebra has only integer dimensions (critical exponents) and can be considered as good candidate to describe behaviors of critical systems. On the other hand, the RCFT's have an important propriety, namely solvability.

A CFT is completely solved once we have specified its central charge (the eigenvalue of the central generator of the Virasoro Algebra), the conformal dimensions of its primary fields and the structure constants of the operator algebra. These three parameters represent the central problem in the classification of conformal field theories and the main goal of the *bootstrap approach*, initially developed in the seminal work of Belavin, Polyakov and Zamolodchikov [3]. The bootstrap idea is based on some general requirements, among which the prominent part played by the associativity of the operator product algebra or crossing symmetry of the four point amplitudes (correlation functions) on the plane. However, upon forming radially ordered products of fields inside correlation fonctions, the fields of the theory will constitute a closed associative operator algebra, so that the operator product of primaries can be written as:

$$\Phi_{\mathbf{I}}\left(z,\overline{z}\right)\cdot\Phi_{\mathbf{J}}\left(w,\overline{w}\right)=\sum_{\mathbf{K}}C_{\mathbf{IJK}}\left(z-w\right)^{h_k-h_i-h_j}\left(\overline{z}-\overline{w}\right)^{\overline{h}_k-\overline{h}_i-\overline{h}_j}\left[\Phi_{\mathbf{K}}\left(w,\overline{w}\right)+\ldots\right]. \quad (1)$$

where $I\equiv\left(i,\overline{i}\right)$ indicate different fields with $i\left(\overline{i}\right)$ representing the contributions (*vertices*) of the holomorphic (antiholomorphic) sectors to physical fields. h_i and $\overline{h}_i$ are conformal dimensions of $\Phi_{\mathbf{I}}$, the ellipsis stands for terms involving descendant fields. $C_{\mathbf{IJK}}$ are complex numbers, known as structure constants of the operator algebra. Let us now consider the four points function

$$G_{\mathbf{IJKL}}\left(z,\overline{z}\right)=\left\langle\Phi_{\mathbf{I}}\left(0\right)\cdot\Phi_{\mathbf{J}}\left(z,\overline{z}\right)\cdot\Phi_{\mathbf{L}}\left(1\right)\cdot\Phi_{\mathbf{K}}\left(\infty\right)\right\rangle. \quad (2)$$

of primary fields by applying the (OPA), on the first two and the last two fields. But using associativity of the (OPA), one could also form products in different order. These different descriptions of the four point correlation G must yield the crossing symmetry relation for the four points functions:

$$G_{\mathbf{IJ}}^{\mathbf{KL}}\left(z,\overline{z}\right)=G_{\mathbf{IL}}^{\mathbf{JK}}\left(1-z,1-\overline{z}\right)=z^{-2h_l}\ \overline{z}^{-2\overline{h}_l}G_{\mathbf{IK}}^{\mathbf{JL}}\left(z^{-1},\overline{z}^{-1}\right). \quad (3)$$

Equivalently, after performing the (OPA), this last relation can be expressed more explicitly as:

$$\begin{aligned}&\sum_{(p,\overline{p})}C_{\mathbf{IJP}}C_{\mathbf{LMP}}\mathcal{F}_{ij}^{lm}\left(p\mid z\right)\overline{\mathcal{F}}_{\overline{i}\overline{j}}^{\overline{l}\overline{m}}\left(\overline{p}\mid\overline{z}\right)\\&\qquad=\sum_{(q,\overline{q})}C_{\mathbf{ILQ}}C_{\mathbf{JMQ}}\mathcal{F}_{lj}^{im}\left(q\mid 1-z\right)\overline{\mathcal{F}}_{\overline{l}\overline{j}}^{\overline{i}\overline{m}}\left(\overline{q}\mid 1-\overline{z}\right).\end{aligned} \quad (4)$$

This equation is called Bootstrap equation where $\mathcal{F}_{ij}^{lm}\left(p\mid z\right)$ denote the conformal blocs. The conformal blocs represent the holomorphic or antiholomorphic pieces in correlation functions. Bootstrap equation is the master equation in conformal field theories. This equation contains all informations required for a complete classification of these theories. Unfortunately, it is very difficult to solve it in practice and other reformulations of classification problem have been proposed.

The most popular of these reformulations is based on modular geometry. In principle, we hope to construct a conformal field theory consistent in various geometries, with various boundary conditions. In particular, the consistent formulation of a conformal theory on the torus has proved to be extremely restrictive on the operator content, i.e. determine how chiral vertices combine to form a physical field. Consistence means, among other things, that the periodic partition function must take the same value on conformally equivalent tori: it must be modular invariant. The partition function takes the form

$$Z\left(\tau,\overline{\tau}\right)=\sum_{i}\chi_{(i)}\left(\tau\right)\ N_{i\overline{i}}\ \chi_{\left(\overline{i}\right)}^{*}\left(\overline{\tau}\right).$$

where $\chi_{(i)}$ is the character of the lowest-weight representation (i) and $N_{i\,\bar{i}}$ are positive integers which determine how many times the two representations (i) and $(\bar{i})$ of the chiral algebra couple with each other. τ is the modular parameter of the torus. Modular invariance implies that $Z(\tau)$ is invariant under transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$.

The program of finding the solutions of modular invariant partition fonctions was particularly developed in the case of Kac-Moody algebras. It is shown that solutions can be arranged in three broad categories [4]:

1. *Diagonal series.* $N_{i\,\bar{i}} = \delta_{i\bar{i}}$. They are often designed as members of the A series.
2. *Complementary series.* They are nondiagonal series and are often designed as members of the D series. These solutions are also called invariant currents solutions.
3. *Exceptional series.* They are nondiagonal series which occur at certain levels. They are often designed as members of the E series.

We aim in this work to present a formulation which will permit to make a new step in the resolution of the bootstrap equation. This formulation consists in the construction of the so-called non-chiral fusion rules. Initially, the term fusion rules was used to express how the chiral vertices (i) and (j) combine in the (OPA) [5]. In practice, to describe fusion rules, we associate to each primary field $\Phi_{\mathbf{I}}$ an abstract object (i) , and introduce an abstract multiplication $(*)$ by:

$$(i) * (j) = \sum_k \mathcal{N}_{ij}^k \, (k) \, . \tag{5}$$

Where $\mathcal{N}_{ij}^k$ (fusion rules coefficients) give the number of distinct ways in which the representation (k) occurs in the fusion of two fields transforming according to the representations (i) and (j) respectively. This process of fusion corresponds to considering only the holomorphic, or only antiholomorphic part of the operator product algebra of two physical operators. This description is incomplete, however, because the holomorphic part (vertex) of the physical field is not physical. Its correlation functions (conformal blocs) are multivalued. So, the physical correlation is constructed when the two pieces are combined together appropriately. This is done by considering the requirement that the conformal blocs in the two sectors combine into single-valued (monodromy invariant) correlation functions.

The construction of monodromy invariant correlation functions is simple in principle for the A series models. In this case, the conformal blocs in the two sectors are complex conjugate so that diagonal terms alone survive the monodromy constraint. This last consideration was first developed by Dotsenko and Fateev [6] for the case of correlation fonctions with spinless fields. Nevertheless, for the non-diagonal D (E) series models the conformal blocs are not complex conjugate and the problem is far more complicated.

By definition the non-chiral fusion rules determine the operator content of the operator algebra, i.e. they determine which fields are present in the product (fusion) of two (physical) fields. The non-chiral fusion rules permit to avoid the monodromy problem and determine the combination of the conformal blocs present in the correlation function.

Initially, the idea of the construction of the non-chiral fusion rules was introduced for the case of the minimal[1] and unitary models [7] [8]. We find that this idea may be generalized to the case of D series models in general RCFT. Our ideas are in the primary state, so a more rigorous and general development must be envisaged.

2 Non-chiral fusion rules

2.1 Basis of construction

By definition, the non-chiral fusion rules determine the operator content of the operator algebra. Thus, the fusion of the two fields $\Phi_{\mathbf{I}}$ and $\Phi_{\mathbf{J}}$ produces the field $\Phi_{\mathbf{K}}$ if and only if the structure constant $C_{\mathbf{IJK}}$ is non-null. If we note the fusion operation by $(*)$ the fusion rules are formally given by:

$$\begin{aligned} (i,\bar{i}) * (j,\bar{j}) &= \sum_{\mathbf{K}} \mathcal{N}_{\mathbf{IJ}}^{\mathbf{K}} \;\; (k,\bar{k})\,; \\ \mathcal{N}_{\mathbf{IJ}}^{\mathbf{K}} \neq 0 &\Leftrightarrow C_{\mathbf{IJK}} \neq 0. \end{aligned} \tag{6}$$

For the determination of the non-chiral fusion rules we use some considerations, imposing strict constraints which we judge sufficient for the determination of these rules. The first constraint lays on the compatibility of the non-chiral fusion rules and the fusion rules (chiral)[2]. In fact, from the bootstrap equation we can see that:

$$C_{\mathbf{IJK}} \neq 0 \Rightarrow \left\{ \begin{array}{l} (i) * (j) \to (k) \\ (\bar{i}) * (\bar{j}) \to (\bar{k}) \end{array} \right. \tag{7}$$

In other words:

$$\mathcal{N}_{\mathbf{IJ}}^{\mathbf{K}} \neq 0 \Rightarrow \mathcal{N}_{ij}^{k} \neq 0 \text{ and } \mathcal{N}_{\bar{i}\bar{j}}^{\bar{k}} \neq 0. \tag{8}$$

[1]The minimal models are the rational models in the case where the chiral algebra is limited to the Virasoro algebra.

[2]This last contraint is somewhat compatibile with the naturality theorem [13].

The second consideration, "obvious", consists on imposing the compatibility of the non-chiral fusion rules with the operator content determined by the modular constraint:

$$\left(i,\bar{i}\right)*\left(j,\bar{j}\right)\rightarrow\left(k,k'\right)\quad\text{if } k'=\overline{k}. \tag{9}$$

The third important fact used in the construction of the non-chiral fusion rules, less obvious than the precedent ones, lays on symmetry considerations. In fact, the first two constraints: the fusion and modular are not completely independent. The most known link between these two constraints is expressed through Verldine's famous formula [9]. The best way to exploit this link and to construct the modular invariant partition functions from the fusion rules is based on the formulation called *"simple currents construction"* . This later formulation leads to the understanding of the particular structure of modular solution of D series.

2.2 Simple currents construction

A simple current (j) is a primary field (chiral part) for which the fusion rules (chiral) with (i) $(j{*}i)$ give just one non-vanishing fusion coefficient $\mathcal{N}_{ji}^{k}$, so that $(j)*(i)=(k)$ [10]. Due to the associativity of the fusion product, the fusion of two simple currents is again a simple current. Simple currents thus form an abelian group under the fusion product, this group has been termed the *center of the theory.* Since the number of primary fields is finite in a rational theory, the number of simple currents is finite. This implies that simple currents are unipotent, i.e. there must be an integer N so that $j^N=1$. The smallest integer N with this property is called the order of the simple current. Furthermore and by rationality, on all other primary fields (i) of the theory, there must be a smallest positive integer d so that $(j)^d*(i)=(i)$. By associativity, we can show that d must be a divisor of N. If a simple current takes a field into itself, we call it a *fixed point* of that current. We see that any simple current organizes the primary fields into orbits

$$[(i)]=\left\{(i)\,,(j*i)\,,\left(j^2*i\right)\ldots\left(j^{d-1}*i\right)\right\}. \tag{10}$$

2.2.1 Monodromy charge

The presence of the simple currents in a conformal theory permits not only to organize the fields in orbits but also to express a symmetry through a conservation of monodromy charge. The monodromy charge $Q\left(i\right)$, defines the monodromy property of the (OPA) (1) of a primary field (i) with a simple current (j)

$$\left(i\right)\left(z\right)\cdot\left(j\right)\left(w\right)\sim\left(z-w\right)^{-Q(i)}\left(i*j\right)\left(w\right). \tag{11}$$

So that:

$$Q\left(i\right)\equiv h_j+h_i-h_{(j*i)}\left[1\right]. \tag{12}$$

This definition makes sense only modulo integers ($\equiv [1]$) because, in general, one cannot be sure about the presence of leading pole in operator product. If Q is non-integer, this means that $(i)(0)$ creates a branch cutt in $(j)(z)$. This last definition imposes that the monodromy charge must be additive under the operator product, i.e.

$$Q(i \cdot k) = Q(i) + Q(k). \tag{13}$$

It follows that all terms in the OPA of two fields *must have the same charge*. Using these properties of monodromy charge and the fact that the only non-vanishing one point function is the identity field function, a correlation function $\langle i_1 i_2 \dots i_n \rangle$ is non-vanishing if

$$\sum_n Q(i_n) = Q(1) = 0. \tag{14}$$

This relation expresses the conservation charge condition (modulo integers) in all correlation functions.

Finally, if we denote the charge associated to a simple current j by Q, then obviously the charge associated to j^d is equal to dQ, as usual modulo integer. This implies that simple currents of order N $\left(j^N = 1\right)$ have a charge equal to an integer, and therefore

$$Q(j) \equiv \frac{r}{N}[1]. \tag{15}$$

r is defined modulo N.

2.2.2 Modular invariant partition functions

By conservation of monodromy charge we have just seen that the presence of simple currents in conformal field theory implies the existence of a discrete symmetry, whose generator is $e^{i2\pi Q(j)}$. This important result can be used to find a solution of modular invariant partition function. The most useful way to do this is to proceed with an *Orbifold-like method* [10] [11]. We start with some known partition function (diagonal solution) and mod it out by discrete group Γ. We shall take Γ to be a discrete group symmetry of a simple current. For simplicity, we restrict ourselves to a current with an integer prime order and cyclic center $\left\{1, j, j^2, \dots, j^{N-1}\right\} \cong Z_N$. In this case the modular invariant partition function is so that [10]:

$$Z = \sum_{i(Q(i)=0)} \left\{ \chi_{(i)}^2 + \sum_{p=1}^{N} \chi_{j^p * i} \overline{\chi}_{j^{N-p} * i} \right\}, \quad Q(j) \neq 0\,[1]; \tag{16}$$

$$Z = \sum_{i(Q(i)=0)} \frac{1}{N} \left| \chi_i + \chi_{j*i} + \chi_{j^2*i} + \dots + \chi_{j^{N-1}*i} \right|^2, \; Q(j) \equiv 0\,[1] \tag{17}$$

The first type of modular invariant partition function is called automorphism [12] invariant type and the second one an integer spin invariant. This last type can be always regarded as a diagonal invariant of larger chiral algebra than originally considered [13]. In this last case, the characters appearing in the same term as the identity (character) have an integer spin and correspond to the extra currents (Noether currents) that extend the algebra. Now, we exploit modular invariant partition functions forms to construct the non-chiral fusion rules.

2.3 The non-chiral fusion rules

2.3.1 The automorphism invariant cases

The operator content of the automorphism invariant cases (16) can be structured in sets:

$$A_k = \left\{ \left(J^k i, J^{N-k} i \right) / Q\left(i\right) = 0 \right\}.$$

Where A_0 designates the scalar fields. To construct the non-chiral fusion rules, let us consider two fields $\left(j^k i, j^{N-k} i\right)$ and $\left(j^p i', j^{N-p} i'\right)$ of A_k and A_p, respectively. By application of our algorithm, we start with the fusion rules. Due to the charge conservation in fusion rules we get:

$$\begin{aligned} \left(j^k i\right) * \left(j^p i'\right) &= j^{k+p} \left(i * i'\right) \to j^{k+p} k, \ Q\left(k\right) = 0; \\ \left(j^{N-k} i\right) * \left(j^{N-p} i'\right) &= j^{N-(k+p)} \left(i * i'\right) \to j^{N-(k+p)} k, \ Q\left(k\right) = 0. \end{aligned}$$

The compatibility with the operator content implies that:

$$\left(j^k i, j^{N-k} i\right) * \left(j^p i', j^{N-p} i'\right) \to \left(j^{p+k} k, j^{N-(p+k)} k\right).$$

so that finally we conclude:

$$A_k * A_p = A_l, \ \ l \equiv p + k \left[N\right]. \tag{18}$$

The non-chiral fusion rules have then a structure of a Z_N grading where N is the order of the simple current considered in the construction of modular invariant partition function.

For scalar fields we have:

$$A_0 * A_0 = A_0. \tag{19}$$

We see that scalar fields form a closed sub-algebra in the (OPA). Therefore we point out here an important fact namely that Dotsenko and Fateev monodromy invariant correlation partition functions with spinless fields have been realized before modular constraint development. If correlation fields are spinless, conformal blocs are complex conjugate and therefore a "minimal coupling" is realized by diagonal combination blocs (monodromy invariant). So that we can conjecture that in general cases*, scalar fields form a sub-algebra in the OPA* (19).

The structure of Z_N grading of the non-chiral fusion rules (18) obtained in the automorphism invariant cases indicates an important fact, namely the conservation of a charge (to be not confused with the monodromy charge) in non-chiral fusion rules. In fact, if we assign to each field in a set (A_p) a charge $e^{\frac{2\pi i p}{N}}$, we see that this charge is conserved in non-chiral fusion rules. This observation is very important particularly in the construction of the non-chiral fusion rules in the integer invariant cases.

2.3.2 The integer invariant cases

In these cases, the problem of the construction of the non-chiral fusion rules is less evident than in the automorphism invariant cases. The origin of difficulties arises principally from the fact that if $Q(j) = 0$, then the charge assigned to $(j^p i)$ with $Q(i) = 0$ is also zero:

$$Q(j^p i) = Q(i) + pQ(j) = 0.$$

So that for each (i) $(Q(i) = 0)$, we can find p and (k) $(Q(k) = 0)$ such that:

$$j^p k = i. \tag{20}$$

As a consequence, the operator content of the invariant integer cases (17) can be structured in sets:

$$A_k = \left\{ \left(i, j^k i\right) / Q(i) = 0 \right\}. \tag{21}$$

From these structures and from the particularity (20), we can easily see that, if we proceed like in automorphism cases, $A_0 * A_0$ will be different from A_0. The fusion of two scalar fields can produce non-scalar fields which are not compatible with our conjecture (19). This is the first problem. The second problem is the presence of fixed point fields of a current (j): $(j) * (i) = (i)$ is possible provided that $Q(j) = 0$.

Since the fixed point of (j) is also a fixed point of $\left(j^k\right)$, the fields $\left(i, j^k i\right) \in A_k$ are equal to (i, i). So, the existence of fixed point chiral field (i) implies the existence of N copies of the same scalar field (i, i) in the operator content and we are not able to predict the behavior of the different copies of this field in the (OPA).

The existence of multiple copies (N copies) of the same primary field $\{\Phi_p, p = 0 \ldots N-1\}$ implies, by the correspondence fields-states principle, that the corresponding lowest-weight state (ground state vector) is degenerate. As in [7] [8], this observation shows the existence of a discrete symmetry (Z_N) of ground states such that:

$$Z_N\left(\Phi_p\right) = e^{\frac{i2\pi p}{N}}\Phi_p. \tag{22}$$

Thus the symmetry (Z_N) enables one to separate the contributions of degenerate field copies, by imposing the compatibility of non-chiral fusion rules with the action of this symmetry. To do this, we must determine the action of (Z_N) in the other fields of the model. For this construction we use an heuristic argument, namely that the fields of the same structure (fields of the same set A_p) must have the same behavior under Z_N. Observing that each copy (Φ_p) of a degenerate field comes in fact from a set A_p, we can conclude that:

$$Z_N\left(\Phi\right) = e^{\frac{i2\pi p}{N}}\Phi,\ \ \Phi \in A_p. \tag{23}$$

Finally, the compatibility of the non-chiral rules with the action of the Z_N symmetry (22, 23) implies a Z_N-grading (like in the automorphism cases) of these rules. The conservation of a charge in non-chiral fusion rules leads us to think that associating a simple current field interpretation would provide our construction with a more rigorous argument.

Current structure of non-chiral fusion rules :

In the integer invariant cases, the simple current (j) is present as chiral part of some physical fields, namely:

$$(j^p, j^p) \in A_0, \quad (1, j^p) \in A_p. \tag{24}$$

These fields may be seen as powers of (j,j) and $(1,j)$. Their action on a field $(i, j^p i)$ $\in A_p$ is determined by our algorithm witch leads to

$$\begin{aligned} (j,j) * (i, j^p i) &= \left(k, j^{p+1}k\right) \in A_p, \quad k = ji; && (25) \\ (1,j) * (i, j^p i) &= \left(i, j^{p+1}i\right) \in A_{p+1}. && (26) \end{aligned}$$

We see that fields (j,j) and $(1,j)$, generated by a simple current (j), are simple currents fields of non-chiral fusion rules. The simple current structure associated to the field $(1,j)$ (26) is particularly interesting because it permits to establish a correspondence between fields in different sets A_p and hence can be used to show the grading structure of non-chiral fusion rules.

However, by using our remark that each component $\{\Phi_p\}_{p=0}^{p=N-1}$ of a degenerate field comes from a set $\{A_p\}_{p=0}^{p=N-1}$, we conjecture that:

$$(1,j) * \Phi_p = \Phi_{p+1}. \tag{27}$$

Since $(1,j)^N = \left(1, j^N\right) = (1,1)$, (27) is a pure reinterpretation of the Z_N symmetry action (22). Combining our first conjecture namely that scalar fields form a sub-algebra in (OPA) and the simple current field action of $(1,j)$ (26), we can deduce the Z_N grading structure of the non-chiral fusion rules law (18).

3 Conclusion

In this work, we have developed a method to construct the non-chiral fusion rules in general rational conformal models of D-like series. This method is based on three considerations:

1. Compatibility with fusion rules;
2. Compatibility with the operator content of the models;
3. Symmetry considerations.

This last point is presented through the currents construction of modular invariant D like series. We have restricted ourselves to currents with integer prime order. The non-chiral fusion rules proposed have a structure of Z_N -grading.

We have conjectured first, by monodromy considerations, that scalar fields form a closed sub-algebra of non-chiral fusion rules. This last conjecture is a direct consequence of our construction method in automorphism invariant cases. Nonetheless, in the integer invariant cases establishing this conjecture is less evident. To overcome difficulties appearing in these last cases, we have used an heuristic argument based essentially on cyclic symmetry (Z_N) ground state structure. This symmetry is assigned to simple current field construction of non-chiral fusion rules. This current field is nothing else that the current of the *extended algebra*. The action of this current on degenerate components field is our second conjecture. It traduces the cyclic structure (Z_N) of the ground state. By exploiting this last statement based on currents extended algebra, we hope to give a more rigorous approach and hence demonstrate our two conjectures.

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